Journal of Statistical Physics, Vol. 66, Nos. 5/6, 1992

Determination of Fixed Points and Shift Cycles for Nearest Neighbor Cellular Automata

Burton Voorhees[1]

Received March 18, 1991; final June 21, 1991

One-dimensional nearest neighbor cellular automata defined over $\mathbf{Z}_2$ are characterized in terms of a set of eight nonadditive basis operators which act on the automaton state space. Every evolution rule for such automata can be expressed as an operator which is a direct sum of the basis operators. This approach allows decomposition of automata rules into additive and nonadditive parts. As a result, it is simple to determine fixed points (those states for which the rule reduces to the identity), and shift cycles (sets of states on which the rule reduces to a shift). Sets of states on which any given nearest neighbor automaton reduces to an identity or a shift are characterized. This allows us to obtain some results on the entropic properties of nonadditive automata, although these are not nearly so complete as results obtained for additive automata.

KEY WORDS: Cellular automata; fixed points and shift cycles; entropy reduction.

1. INTRODUCTION

One of the basic questions which arises in the study of cellular automata is to determine their fixed points and cycling behavior. As it turns out, cycling behavior is particularly difficult to characterize and even such important parameters as maximum cycle period are, in general, unknown, although upper bounds have been obtained.[1-3]

For finite, one-dimensional, nearest neighbor automata the question of maximum period has usually been posed in the form: "given a particular automaton rule, the number of cells, and specified (usually periodic) boundary conditions, what is the maximum cycle period?" With this approach, it has been shown[1-3] that for an additive automata defined over

[1] Faculty of Arts & Sciences, Athabasca University, Athabasca, Alberta, Canada TOG 2RO.

0022-4715/92/0300-1397$06.50/0

$\mathbf{Z}_p$ (p prime), with n cells and periodic boundary conditions, the maximum cycle period $b(n)$ divides the number $b^*(n)$ defined by

$$b^*(n) = \min \begin{cases} p^k - p^m \\ (p^l - p^m)n_0 \end{cases} \tag{1.1}$$

where $n = p^m n_0$, p does not divide n_0, and k, l are the smallest integers such that, respectively, $p^k \equiv p^m \bmod(n)$ and $p^l \equiv -p^m \bmod(n)$. Note that l does not always exist (e.g., for $p = 2$ and $n = 7$).

It is also known[1,2] that the periods of all cycles divide the maximum period $b(n)$. In many cases it is true that $b(n) = b^*(n)$, but there are notable exceptions (e.g., see the table of maximum cycle periods in ref. 1). Jen[2] has shown that these exceptions are a result of the fact that certain values of n allow "anomalous shifts" which act to reduce maximum cycle period.

In an excellent paper Jen[2] has also turned the maximum period question around, asking it in the form: "given a number $b(n)$, for what values of n will this be the cycle period for a specified automaton rule?" She provides answers to this question in terms of recursion relations over finite fields. In addition, she shows the fundamental role of shifts in automata cycles: if the state space for a one-dimensional automaton with n cells and periodic boundary conditions is denoted E_n, then the automaton evolution rule has a natural expression as an operator Q: $E_n \to E_n$, and the automaton will be denoted (Q, E_n). Taking σ^{-1} as the right shift operator on E_n, we can state Jen's result on the role of shifts as follows.

Theorem 1.[3] A state $\mu \in E_n$ lies on a cycle of an additive cellular automaton (Q, E_n) if and only if there exist r, s such that $Q^r(\mu) = \sigma^{-s}(\mu)$. [We note that $r = b(n)$, $s = n$ is a possibility.]

The fundamental role of shifts in the construction of cycles for cellular automata raises the question of shift cycles; i.e., cycles on which an automaton rule acts as a shift. In this paper an approach to cellular automata in terms of a set of "orthogonal" operators which span the automata rule space is used to determine and classify fixed points and shift cycles for all nearest neighbor cellullar automata with periodic boundary conditions, defined over $\mathbf{Z}_2$. We also indicate how this approach may be extended to non-nearest-neighbor automata, to automata defined over $\mathbf{Z}_p$ (p prime), and to higher-dimensional automata.

2. FINITE NEAREST NEIGHBOR CELLULAR AUTOMATA WITH PERIODIC BOUNDARY CONDITIONS

Jen[2] terms these automata cylindrical, since their time evolution is most naturally represented on a cyclinder. In ref. 3 a detailed study of these automata has been carried out, based on a formalism which is briefly reviewed here.

If the automaton rule is such that the value in cell i at time $t+1$ is determined only on the basis of the values in cells $i-1$, i, and $i+1$ at time t, then the automaton follows a nearest neighbor rule. Denote this automaton (Q, E_n). If, for all $\mu, \mu' \in E_n$, $Q(\mu+\mu') = Q(\mu) + Q(\mu')$, the authomaton rule is said to be additive.

If μ_i denotes the ith component (i.e., the value in cell i) of a state μ, then the general component form for the action of the operator representing an additive nearest neighbor rule is

$$\mu_i(t+1) = [Q(\mu(t))]_i = x\mu_{i-1}(t) + y\mu_i(t) + z\mu_{i+1}(t) \tag{2.1}$$

where $x, y, z \in \mathbf{Z}_2$ and all sums and products are reduced mod(2), while all component indices are reduced mod(n).

The state consisting of all O's will be indicated by $\mathbf{0}$ and the state consisting of all 1's by $\mathbf{1}$.

Equation (2.1) defines eight distinct additive operators, including the trivial zero operator which maps all states to $\mathbf{0}$. If I, σ, and σ^{-1} denote, respectively, the identity, left shift, and right shift on E_n, then (2.1) is equivalent to the operator equation

$$Q = x\sigma^{-1} + yI + z\sigma \tag{2.2}$$

The explicit forms for the nontrivial additive nearest neighbor operators are given in Table I, using the labeling scheme initiated by Wolfram.[4]

In order to include nonadditive operators, we define a set of eight nonadditive operators on E_n corresponding to the automata labeled 128, 64, 32, 16, 8, 4, 2, and 1 in Wolfram's catalogue. Definitions of these operators are provided in Table II.

Table I. Nearest Neighbor Additive Cellular Automata Over $\mathbf{Z}_2$

(x, y, z)	Operator	Component Form	Rule
(0, 0, 0)	0^*	$[0^*(\mu)]_i = 0$	0
(1, 0, 0)	σ^{-1}	$[\sigma^{-1}(\mu)]_i = \mu_{i-1}$	240
(0, 1, 0)	I	$[I(\mu)]_i = \mu_i$	204
(0, 0, 1)	σ	$[\sigma(\mu)]_i = \mu_{i+1}$	170
(1, 1, 0)	D^-	$[D^-(\mu)]_i = \mu_{i-1} + \mu_i$	60
(0, 1, 1)	D	$[D(\mu)]_i = \mu_i + \mu_{i+1}$	102
(1, 0, 1)	δ	$[\delta(\mu)]_i = \mu_{i-1} + \mu_{i+1}$	90
(1, 1, 1)	Δ	$[\Delta(\mu)]_i = \mu_{i-1} + \mu_i + \mu_{i+1}$	150

Table II. Basis Operators for Canonical Representation

Automaton	Operator	Rule
128	χ	$[111\} \to 1$, all others $\to 0$
64	β^+	$\{110\} \to 1$, all others $\to 0$
32	Θ	$[101\} \to 1$, all others $\to 0$
16	η^-	$\{100\} \to 1$, all others $\to 0$
8	β^-	$\{011\} \to 1$, all others $\to 0$
4	ι	$[101\} \to 1$, all others $\to 0$
2	η^+	$\{001\} \to 1$, all others $\to 0$
1	κ	$[000\} \to 1$, all others $\to 0$

Since whenever a site maps to 1 under one of these operators it maps to 0 under the remaining seven, there is no interference, and every operator representing a nearest neighbor rule over $\mathbf{Z}_2$ can be uniquely expressed as a direct sum of these eight operators. That is, the operators of Table II provide a basis for the nonlinear algebra of operators defined by the set of nearest neighbor rules over $\mathbf{Z}_2$. Expression of an operator Q in terms of these basis operators will be called the canonical representation of Q.

To determine the canonical representation of an operator, its numeric label is written in powers of 2 and the appropriate substitutions are made from Table II. For example, the canonical representation of rule 176 is obtained via $176 = 128 + 32 + 16 = \eta^- + \chi + \theta$. For further properties of these basis operators the reader is referred to ref. 3.

The main results of this paper are based on the canonical representations of the seven nontrivial additive operators. These are given in Table III.

Canonical representations, both additive and nonadditive, can be added with coefficients reduced mod(2). For example,

$$D = I + \sigma = (\beta^+ + \beta^- + \chi + \iota) + (\beta^- + \eta^+ + \theta + \chi) = \beta^+ + \eta^+ + \theta + \iota$$

This means that every nearest neighbor rule can be decomposed into an additive and a nonadditive part, although it turns out that this decomposition is not generally unique. We will be particularly interested in those cases in which the additive part of a rule is the identity, or a shift.

If Q^* is a nonadditive operator with decomposition $Q^* = Q + F$, where Q is an additive operator, and if Q^* reduces to Q one some nonempty subset of E_n-$\{\mathbf{0}, \mathbf{1}\}$, the decomposition will be called legal. If the canonical representation of Q^* contains κ, then Q^* will be said to be generative. For simplicity, considerations are restricted to nongenerative rules.

Table III. Canonical Representation of Additive Operators

Automaton	Operator	Rule
0	0^*	All map to 0
170	$\sigma=\beta^- +\eta^+ +\Theta+\chi$	[001, 001, 101, 111} → 1 {110, 100, 010, 000} → 0
240	$\sigma^{-1}=\beta^+ +\eta^- +\Theta+\chi$	[100, 100, 101, 111} → 1 {011, 001, 010, 000} → 0
204	$I=\beta^+ +\beta^- +\chi+\iota$	[011, 110, 111, 010} → 1 {100, 001, 101, 000} → 0
102	$D=\beta^+ +\eta^+ +\Theta+\iota$	[110, 001, 101, 010} → 1 {011, 100, 111, 000} → 0
60	$D^-=\beta^- +\eta^- +\Theta+\iota$	[011, 100, 101, 010} → 1 {110, 001, 111, 000} → 0
90	$\delta=\beta^+ +\beta^- +\eta^+ +\eta^-$	{110, 011, 001, 100} → 1 {101, 111, 010, 000} → 0
150	$\Delta=\eta^+ +\eta^- +\chi+\iota$	[001, 100, 111, 010} → 1 {110, 011, 101, 000} → 0

Canonical representations, together with their legal decompositions, are listed in the Appendix for all nongenerative nearest neighbor rules over $\mathbf{Z}_2$.

3. FIXED POINTS AND SHIFT CYCLES

Suppose that Q^* is a nongenerative, nonadditive automaton with legal decomposition $Q^*=Q+F(\beta^\pm, \eta^\pm, \theta, \chi, \iota)$ with Q being I, σ, or σ^{-1}. We ask the question of what constraints must be satisfied by a state μ if the equation $Q^*(\mu)=Q(\mu)$ is to be true. That is, we look for the subset of E_n which maps to $\mathbf{0}$ under F.

For given Q^* the set of all legal decompositions partitions into four subsets. The first consists of those automata for which Q^* reduces to Q only on a highly restricted subset of E_n. These decompositions will be called weakly legal. The nonadditive part F in a weakly legal decomposition will contain combinations of basis operators which include $\beta^\pm+\eta^\pm$, $\eta^\pm+\chi$, or $\eta^\pm+\chi+\iota$. The full set of weakly legal nongenerative operator forms are listed in Table IV.

The forms listed in Table IV reduce to Q either on states $\underline{01}$ and $\underline{10}$ (underlining indicates repetition) (subset a; requiring n even); on states

Table IV. Weakly Legal Nongenerative Operator Forms

(a) Weakly legal forms which reduce to Q only on $\underline{10}$ and $\underline{01}$:
$Q+\beta^{+}+\eta^{-}$
$Q+\beta^{-}+\eta^{+}$
$Q+\beta^{+}+\eta^{+}$
$Q+\beta^{-}+\eta^{-}$
$Q+\beta^{+}+\beta^{-}+\eta^{+}$
$Q+\beta^{+}+\beta^{-}+\eta^{-}$
$Q+\beta^{+}+\eta^{+}+\eta^{-}$
$Q+\beta^{-}+\eta^{+}+\eta^{-}$
$Q+\beta^{+}+\beta^{-}+\eta^{+}+\eta^{-}$
$Q+\beta^{+}+\eta^{-}+\chi$
$Q+\beta^{-}+\eta^{+}+\chi$
$Q+\beta^{+}+\eta^{+}+\eta$
$Q+\beta^{-}+\eta^{+}+\chi$
$Q+\beta^{+}+\beta^{-}+\eta^{+}+\chi$
$Q+\beta^{+}+\beta^{-}+\eta^{-}+\chi$
$Q+\beta^{+}+\eta^{+}+\eta^{-}+\chi$
$Q+\beta^{-}+\eta^{+}+\eta^{-}+\chi$
$Q+\beta^{+}+\beta^{-}+\eta^{+}+\eta^{-}+\eta$

(b) Weakly legal forms which reduce to Q only on $\underline{110}$, $\underline{101}$, and $\underline{011}$:
$Q+\eta^{+}+\chi+\iota$
$Q+\eta^{-}+\chi+\iota$
$Q+\eta^{+}+\eta^{-}+\chi+\iota$

(c) Weakly legal forms which reduce to Q only on states with single or double 1's separated by isolated 0's:
$Q+\eta^{+}+\chi$
$Q+\eta^{-}+\chi$
$Q+\eta^{+}+\eta+\chi$

$\underline{110}$, $\underline{101}$, and $\underline{011}$ (subset b; requiring $n=3m$); or on states consisting of single or double 1's separated by isolated 0's (subset c).

The remaining three subclasses of Q-decompositions are given in Table V.

Operators in subclass A reduce to Q on the set of states having only isolated ones (those involving $\beta^{\pm}$), or only isolated zeros (those involving $\eta^{\pm}$). Operator forms in subclass B reduce to Q on states having only isolated ones separated by two or more zeros (those involving $\beta^{\pm}$), or only isolated zeros separated by two or more ones (those involving $\eta^{\pm}$). The constraints are more complicated for subclass C: no isolated zeros $(Q+\theta)$; no isolated ones $(Q+\iota)$; no isolated zeros or ones $(Q+\theta+\iota)$; only single or double ones $(Q+\chi)$; only single or double ones, and no islated zeros $(Q+\theta+\chi)$; only double ones $(Q+\chi+\iota)$; only double ones, and no isolated zeros $(Q+\theta+\chi+\iota)$.

Table V. Strongly Legal Nongenerative Forms

	Representation	Constraints for linearity in $E_n - \{\mathbf{0}, \mathbf{1}\}$
A.	$Q+\beta^+$	
	$Q+\beta^-$	
	$Q+\beta^+ +\beta^-$	Reduces to Q on states with only isolated 1's
	$Q+\beta^+ +\chi$	
	$Q+\beta^- +\chi$	
	$Q+\beta^+ +\beta^- +\chi$	
	$Q+\eta^+$	
	$Q+\eta^-$	Reduces to Q on states with only isolated 0's
	$Q+\eta^+ +\eta^-$	
B.	$Q+\beta^+ +\Theta$	
	$Q+\beta^- +\Theta$	
	$Q+\beta^+ +\beta^- +\Theta$	Reduces to Q on states with only isolated 1's separated by two or more 0's
	$Q+\beta^+ +\Theta+\chi$	
	$Q+\beta^- +\Theta+\chi$	
	$Q+\beta^+ +\beta^- +\Theta+\chi$	
	$Q+\eta^+ +\iota$	
	$Q+\eta^- +\iota$	Reduces to Q on states with only isolated 0's separated by two or more 1's
	$Q+\eta^+ +\eta^- +\iota$	
C.		Reduces tpo Q on states:
	$Q+\Theta$	...without isolated 0's
	$Q+\chi$	...with only single or double 1's
	$Q+\iota$	...without isolated 1*s
	$Q+\Theta+\chi$	...with single or double 1's, no isolated 0's
	$Q+\Theta+\iota$	...without isolated 0's or 1's
	$Q+\chi+\iota$	...with only double 1's
	$Q+\Theta+\chi\iota$	...with only double 1's and no isolated 0's

For an operator $Q^* = Q + F(\beta^{\pm}, \eta^{\pm}, \theta, \chi, \iota)$ we denote the set of states on which Q^* reduces to Q by $E_n(Q, F)$. Elementary inclusion arguments yield the following result.

Theorem 2. The following conditions hold:

A. $E_n(Q, a\eta^+ + b\eta^-) = \mathbf{1} + E_n(Q, c\beta^+ + d\beta^- + e\chi)$ (a, b & c, d not both 0)

B. $E_n(Q, \theta) = \mathbf{1} + E_n(Q, \iota)$

C. $E_n(Q, \chi+\iota) \cap E_n(Q, \theta+\chi) = E_n(Q, \chi+\iota) \cap E_n(Q, \theta+\iota)$
$$= E_n(Q, \theta+\chi) \cap E_n(Q, \theta+\iota) = E_n(Q, \theta+\chi+\iota)$$

D. $E_n(Q, \theta) \cap E_n(Q, \chi) = E_n(Q, \theta+\chi)$;
$E_n(Q, \theta) \cap E_n(Q, \iota) = E_n(Q, \theta+\iota)$

E. $E_n(Q, \chi) \supset E_n(Q, a\beta^+ + b\beta^- + c\theta + d\chi)$, a, b, c, d not all 0.

Table VI. Legal Identity Representations of Nongenerative Rules

Weakly Legal Identity Representations:

(a) Only 10 and 01 are fixed: $I+\beta^{+}+\eta^{-}$ (156); $I+\beta^{-}+\eta^{+}$ (198); $I+\beta^{+}+\eta^{+}$ (142); $I+\beta^{-}+\eta^{-}$ (212); $I+\beta^{+}+\beta^{-}+\eta^{+}$ (134); $I+\beta^{+}+\beta^{-}+\eta^{-}$ (148); $I+\beta^{+}+\eta^{+}+\eta^{-}$ (158); $I+\beta^{-}+\eta^{+}+\eta^{-}$ (214); $I+\beta^{+}+\beta^{-}+\eta^{+}+\eta^{-}$ (150); $I+\beta^{+}+\eta^{-}+\chi$ (28); $I+\beta^{-}+\eta^{+}+\chi$ (70); $I+\beta^{+}+\eta^{+}+\chi$ (14); $I+\beta^{-}+\eta^{-}+\chi$ (84); $I+\beta^{+}+\beta^{-}+\eta^{+}+\chi$ (6); $I+\beta^{+}+\beta^{-}+\eta^{-}+\chi$ (20); $I+\beta^{+}+\eta^{+}+\eta^{-}+\chi$ (30); $I+\beta^{-}+\eta^{+}+\eta^{-}+\chi$ (86); $I+\beta^{+}+\beta^{-}+\eta^{+}+\eta^{-}+\chi$ (22)

(b) Only 110, 101, and 011 are fixed: $I+\eta^{+}+\chi+\iota$ (75); $I+\eta^{-}+\chi+\iota$ (88); $I+\eta^{+}+\eta^{-}+\chi+\iota$ (90)

(c) Only states with single or double 1's separated by isolated 0's are fixed: $I+\eta^{+}+\chi$ (78); $I+\eta^{-}+\chi$ (92); $I+\eta^{+}+\eta^{-}+\chi$ (94)

Strongly Legal Identity Representations:

(A1) All states with only isolated 1's are fixed: $I+\beta^{+}$ (140); $I+\beta^{-}$ (196); $I+\beta^{+}+\beta^{-}$ (132); $I+\beta^{+}+\chi$ (12); $I+\beta^{-}+\chi$ (68); $I+\beta^{+}+\beta^{-}+\chi$ (4)

(A2) All states with only isolated 0's are fixed: $I+\eta^{+}$ (206); $I+\eta^{-}$ (220); $I+\eta^{+}+\eta^{-}$ (222)

(B1) All states with only isolated 1's separated by at least two 0's are fixed: $I+\beta^{+}+\Theta$ (172); $I+\beta^{-}+\Theta$ (228); $I+\beta^{+}+\beta^{-}+\Theta$ (164); $I+\beta^{+}+\Theta+\chi$ (44); $I+\beta^{-}+\Theta+\chi$ (100); $I+\beta^{+}+\beta^{-}+\Theta+\chi$ (36)

(B2) All states with only isolated 0's separated by at least two 1's are fixed: $I+\eta^{+}+\iota$ (202); $I+\eta^{-}+\iota$ (216); $I+\eta^{+}+\eta^{-}+\iota$ (218)

(C) $I+\Theta$ (236) fixed points have no isolated 0's; $I+\chi$ (76) fixed points have only single or double 1's; $I+\iota$ (200) fixed points have no isolated 1's; $I+\Theta+\chi$ (108) fixed points have only single or double 1's and no isolated 0's; $I+\Theta+\iota$ (232) fixed points have no isolated 0's or 1's; $I+\chi+\iota$ (72) fixed points have only double 1's; $I+\Theta+\chi+\iota$ (104) fixed points have only double 1's and no isolated 0's

It is now possible to classify the fixed point and the single shift behavior of all nearest neighbor cellular automata defined over $\mathbf{Z}_2$.

Table VI lists all nongenerative automata having fixed points contained in $E_n - \{\mathbf{0}, \mathbf{1}\}$.

Table VII provides a similar breakdown for those automata which reduce to right or left shifts. Both shifts are indicated in the same table, since for each σ-representation of an automaton rule there is a conjugate σ^{-1}-representation, obtained by the transformations $\beta^{+} \leftrightarrow \beta^{-}$, $\eta^{+} \leftrightarrow \eta^{-}$. The notation (X, Y) is used to indicate that the automata labeled X and Y are conjugate. That is, exchanging pluses and minuses in the canonical representation is numerically equivalent to exchanging X and Y.

There are several special cases:

1. Automata (184, 226) have decompositions $\sigma^{-1}+\beta^{+}+\beta^{-}=\sigma+\eta^{+}+\eta^{-}$, and $\sigma+\beta^{+}+\beta^{-}=\sigma^{-1}+\eta^{+}+\eta^{-}$, indcicating that for automaton 184 all states having only isolated 1's lie on σ^{-1}-cycles, while all states having only isolated zeros lie on σ-cycles; with conjugate results for automaton 226.

2. Automata (172, 228) have as fixed points all states having only isolated ones separated by two or more zeros, while all states having only isolated zeros separated by two or more ones lie on σ^{-1}-cycles (228) or σ-cycles (172).

3. Automata (202, 216) have as fixed points all states consisting only of isolated zeros separated by two or more ones, while all states consisting of isolated ones separated by two or more zeros lie on either σ^{-1}-cycles (216) or σ-cycles (202). Thus, automata (202, 216) may be viewed as complementary to automata (172, 228) in the sense that fixed points of (202, 216) lie on cycles of (172, 228), and vice versa.

Theorem 3. All automata in Table VIIA–C except (100, 44), (102, 60), (120, 106), (88, 74), (212, 142), and (84, 14) have only single shift cycles.

Proof. We present a proof for the automaton 248. Proof for all other cases follows a similar pattern.

The canonical representation for 248 is $\beta^{+}+\beta^{-}+\eta^{-}+\theta+\chi$. If a state μ has only isolated 1's, then the representation $\sigma^{-1}+\beta^{-}$ indicates that rule 248 acts as a right shift on μ; hence μ must lie on a shift cycle with period $d|n$, where d is the spatial period of μ.

On the other hand, suppose that μ contains a string of ones of the form ...011...110.... After a single iteration of the automaton rule, this

Table VII. Legal Single Shift Representations of Nongenerative Rules

Weakly Legal Single Shift Representations:

(a) Representations which reduce to shift only on $\underline{10}$ and $\underline{01}$:

250	$\sigma+\beta^{+}+\eta^{-}=\sigma^{-1}+\beta^{-}+\eta^{-}$
160	$\sigma+\beta^{-}+\eta^{+}=\sigma^{-1}+\beta^{+}+\eta^{-}$
232	$\sigma+\beta^{+}+\eta^{+}=\sigma^{-1}+\beta^{-}+\eta^{-}$
178	$\sigma+\beta^{-}+\eta^{-}=\sigma^{-1}+\beta^{+}+\eta^{+}$
122	$\sigma+\beta^{+}+\eta^{-}+\chi=\sigma^{-1}+\beta^{-}+\eta^{+}+\chi$
32	$\sigma+\beta^{-}+\eta^{+}+\chi=\sigma^{-1}+\beta^{+}+\eta^{-}+\chi$
104	$\sigma+\beta^{+}+\eta^{+}+\chi=\sigma^{-1}+\beta^{-}+\eta^{-}+\chi$
50	$\sigma+\beta^{-}+\eta^{-}+\chi=\sigma^{-1}+\beta^{+}+\eta^{+}+\chi$
(176, 162)	$\sigma+\beta^{-}+\eta^{+}+\eta^{-}$, $\sigma^{-1}+\beta^{+}+\eta^{+}+\eta^{-}$ (see also $A1$ below)

Table continued

Table VII. *(Continued)*

(b) Representations which reduce to shifts only on $\underline{110}$, $\underline{101}$, and $\underline{011}$:

(102, 60) $\sigma+\beta^{+}+\beta^{-}+\chi+\iota=\sigma^{-1}+\eta^{+}+\eta^{-}+\chi+\iota=D$

$\sigma+\eta^{+}+\eta^{-}+\chi-\iota=\sigma^{-1}+\beta^{+}+\beta^{-}+\chi+\iota=D^{-}$

(44, 100) $\sigma+\eta^{+}+\chi+\iota, \sigma^{-1}+\eta^{-}+\chi+\iota$

(62, 118) $\sigma+\eta^{-}+\chi+\iota, \sigma^{-1}+\eta^{+}+\chi+\iota$

(c) Representations which reduce to shifts only on states with single or double 1's and isolated 0's:

(40, 96) $\sigma+\eta^{+}+\chi, \sigma^{-1}+\eta^{-}+\chi$

(58, 114) $\sigma+\eta^{-}+\chi, \sigma^{-1}+\eta^{+}+\chi$

Strongly Legal Single Shift Representations:

($A1$) Representations which reduce to shifts on states with only isolated 1's:

(234, 248) $\sigma+\beta^{+}, \sigma^{-1}+\beta^{-}$

(162, 176) $\sigma+\beta^{-}, \sigma^{-1}+\beta^{+}$

(106, 120) $\sigma+\beta^{+}+\chi, \sigma^{-1}+\beta^{-}+\chi$

(34, 48) $\sigma+\beta^{-}+\chi, \sigma^{-1}+\beta^{+}+\chi$

($A2$) Representations which reduce to shifts on states with only isolated 0's:

(168, 224) $\sigma+\eta^{+}, \sigma^{-1}+\eta^{-}$

(186, 242) $\sigma+\eta^{-}, \sigma^{-1}+\eta^{+}$

($A3$) Special cases:

(226, 184) $\sigma+\beta^{+}+\beta^{-}=\sigma^{-1}+\eta^{+}+\eta^{-}, \sigma^{-1}+\beta^{+}+\beta^{-}=\sigma+\eta^{+}+\eta^{-}$

(98, 56) $\sigma+\beta^{+}+\beta^{-}+\chi=\sigma^{-1}+\eta^{+}+\eta^{-}+\chi, \sigma^{-1}+\beta^{+}+\beta^{-}+\chi=\sigma+\eta^{+}+\eta^{-}+\chi$

(226 is a left shift on states with only isolated 1's, and a right shift on states with only isolated 0's; and vice versa for 184; 98 is a left shift on states with only isolated 1's, and a right shift on states with single or double 1's and only isolated 0's, and vice versa for 56)

($B1$) Representations which reduce to shifts on states with only isolated 1's separated by at least two 0's:

(202, 216) $\sigma+\beta^{+}+\Theta, \sigma^{-1}+\beta^{-}+\Theta$

(130, 144) $\sigma+\beta^{-}+\Theta, \sigma^{-1+\beta}{}_{+}+\Theta$

(74, 88) $\sigma+\beta^{+}+\Theta+\chi, \sigma^{-1}+\beta^{-}+\Theta+\chi$

(2, 16) $\sigma+\beta^{-}+\Theta+\chi, \sigma^{-1}+\beta^{+}+\Theta+\chi$

(66, 24) $\sigma+\beta^{+}+\beta^{-}+\Theta+\chi, \sigma^{-1}+\beta^{+}+\beta^{-}+\Theta+\chi$

($B2$) Representations which reduce to shift on states with only isolated 0's separated by at least two 1's:

(168, 224) $\sigma+\eta^{+}+\iota, \sigma^{-1}+\eta^{-}+\iota$

(186, 242) $\sigma+\eta^{-}+\iota, \sigma^{-1}+\eta^{+}+\iota$

(230, 188) $\sigma+\eta^{+}+\eta^{-}+\iota, \sigma^{-1}+\eta^{+}+\eta^{-}+\iota$

(C) Reduce to shifts on states with:

(138, 208)	$\sigma+\Theta, \sigma^{-1}+\Theta$	...no isolated 0's
(42, 112)	$\sigma+\chi, \sigma^{-1}+\chi$	...only single or double 1's
(174, 244)	$\sigma+\iota, \sigma^{-1}+\iota$	...no isolated 1's
(10, 80)	$\sigma+\Theta+\chi, \sigma^{-1}+\Theta+\chi$	...single or double 1's, no isolated 0's
(142, 212)	$\sigma+\Theta+\iota, \sigma^{-1}+\Theta+\iota$	...no isolated 0's or 1's
(46, 116)	$\sigma+\chi+\iota, \sigma^{-1}+\chi+\iota$	...only double 1's
	$\sigma+\Theta+\chi+\iota, \sigma^{-1}+\Theta+\chi+\iota$	...only double 1's, no isolated 0's

becomes either ...111...111... or ...011...111.... That is, the length of this string has irreversibly increased by at least one. Thus any μ containing two or more adjacent one's will eventually iterate to the fixed point **1**. ▮

For automata in classes A and B of Table V, there is a simple recursion relation which gives the number of states on which the rule reduces to the additive operator Q. Let $S_n(r; s)$ stand for the subset of E_n consisting of states composed of blocks of exactly r ones separated by blocks of s or more zeros.

Lemma 1. Let $N_n(r; s)$ be the number of elements contained in $S_n(r; s)$. Then, for $1 \leqslant m < r$, $N_n(r; s) = N_n(r-m; s+m)$.

Lemma 2. $N_{n+1}(1; s) = N_n(1; s) + N_{n-s}(1; s) + 1$.

We note that if the roles of 1's and 0's are interchanged in the above lemmas, the results still obtain.

Corollary. For $n > 1$ the number of states in E_n having only isolated ones, or only isolated zeros, is given by $L_n - 1$, where L_n is the nth Lucas number.

Proof. By Lemma 2, $N_{n+1}(1; 1) = N_n(1; 1) + N_{n-1}(1; 1) + 1$. Add one to each side of this equality: $N_{n+1}(1; 1) + 1 = (N_n(1; 1) + 1) + (N_{n-1}(1; 1) + 1)$. This is the Fibonacci recursion relation. We compute $N_1(1; 1) = 0$; $N_2(1; 1) = 2$; hence the sequence of numbers $N_n(1; 1) + 1$ is 1, 3, 4, 7, 11, 18, 29,..., and this is the Lucas sequence. ▮

These results allow enumeration of fixed points, or states on cycles for all those automata in subclasses A and B in Tables VI and VII, and those in subclass C having the form $Q + \chi + \iota$ or $Q + \theta + \chi + \iota$, excepting those automata listed in Theorem 2, which have cycles other then single shift cycles.

4. A FEW REMARKS ON ENTROPY

In several papers the entropy-reducing properties of cellular automata have been studied.[1, 4–6] Roughly, if all automaton states are considered as having equal initial *a priori* probabilities, then as the automaton evolves, certain states will become forbidden. That is, during the time evolution of automata for which not all states lie on cycles, the probability of certain states will become 0, while the probabilities of other states will increase. For example, "Garden of Eden" states, those which have no predecessors unbder the given automaton rule, have probability 0 after a single iteration. In general, if $a(n)$ is the maximum number of iterations required to reach a fixed point or cycle, then, after at most $a(n)$ steps, all remaining accessible states will be fixed points, or lie on cycles.

For additive nearest neighbor automata over $\mathbf{Z}_p$ a complete analysis of entropy-reducing properties, for the normalized measure entropy, has already been given[6]: If Q is such an additive operator, then the normalized measure entropy, defined by

$$S(t) = -(n \log_2 p)^{-1} \sum_{\mu \in E_n} P_t(\mu) \log_2 P_t(\mu) \tag{4.1}$$

where $P_t(\mu)$ is the probability of state μ after t iterations of Q. $S(t)$ satisfies the entropy evolution formula

$$S(t) = \begin{cases} 1 - t\nu(Q')/n, & 0 \leqslant t < a(n) \\ 1 - a(n)\,\nu(Q')/n, & t \geqslant a(n) \end{cases} \tag{4.2}$$

where $a(n)$ is the maximum number of iterations required to reach a cycle (for $n = p^m n_0$ this is usually p^m); Q' is the matrix representation of the operator Q, i.e., the $n \times n$ right circulant matrix with first row $(y, z; 0,..., x)$ with x, y, and z as defined in Eq. (2.1); and $\nu(Q')$ is the nullity of this matrix.

The fractional change in $S(t)$ per time step when $t < a(n)$ is given by $a^{-1}(S(t+1) - S(t)) = -\nu(Q')/na(n)$. Thus, the nullity of the matrix Q' appears as a discrete analogue of the rate of change of entropy. It describes the rate at which the state space "loses dimensions" as the operator is iterated.

Formulas for the nullity of the various nearest neighbor additive operators are given as follows.

Theorem 4. The following relations hold:

$$\nu(D_{(y,z)}) = \nu(D^{-}_{(x,y)}) = \begin{cases} p & (p - z^{-1}y)^n \equiv 1 \bmod(p) \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

$$\nu(\delta_{(x,z)}) = \begin{cases} p^2 & n \text{ even}, \quad (p - z^{-1}x)^n \equiv 1 \bmod(p) \\ p & n \text{ odd}, \quad (p - z^{-1}x)^n \equiv 1 \bmod(p) \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

$$\nu(\Delta_{(x,y,z)}) = \begin{cases} p^2 & A_{n+1} = 1, \quad B_{n+1} = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

where the operators are generalizations of the additive operators defined over $\mathbf{Z}_2$, given in component from by

$$[D_{(y,z)}(\mu)]_i = y\mu_i + z\mu_{i+1}$$
$$[D^{-}_{(y,z)}(\mu)]_i = x\mu_{i-1} + y\mu_i$$
$$[\delta_{(x,z)}(\mu)]_i = x\mu_{i-1} + z\mu_{i+1}$$
$$[\Delta_{(x,y,z)}(\mu)]_i = x\mu_{i-1} + y\mu_i + z\mu_{i-1}$$

and the terms A_i and B_i in (4.5) are defined by taking $A_1 = B_2 = 1$, $A_2 = B_1 = 0$, while for $i \geqslant 3$, with $\{a\}$ the least integer greater than or equal to a, and $\Pi^{(k)_j}$ the jth entry in the kth row of the mod(p) Pascal triangle,

$$\begin{aligned} A_i &= \sum_{j=0}^{\{(i-3)/2\}} \Pi^{(i-j-2)}_{i-2j-2}(p - z^{-1}x)^{j+1}(p - z^{-1}y)^{i-2j-3} \\ B_i &= \sum_{j=0}^{\{(i-2)/2\}} \Pi^{(i-j-1)}_{i-2j-1}(p - z^{-1}c)^{j}(p - z^{-1}y)^{i-2j-2} \end{aligned} \tag{4.6}$$

subject to constraints $A_{n+1} = 1$, $B_{n+1} = 0$.[6]

As a result of this theorem, only certain cellular automata are seen to decrease entropy, namely those for which $\nu\ (Q') \neq 0$. By (4.2) the entropy for these automata, with $n = p^m n_0$ and $a(n) = p^m$, evolves according to the formula

$$S(t) = \begin{cases} t - t\nu(Q')/p^m n_0 & 0 \leqslant t < p^m \\ 1 - \nu(Q')/n_0 & p^m \leqslant t \end{cases} \tag{4.7}$$

and the final value of this entropy is independent of m.

For nonadditive automata, even restricting the field of definition to $\mathbf{Z}_2$, the situation is far more difficult and no general formula for the time evolution of entropy has been found. The reason behind this is that additive automata have the property that all trees rooted on cycles or fixed points of the state transition diagram have the same height.[1] This is not generally true for nonadditive automata, and so case-by-case analysis is required. It is possible, however, on the basis of the results presented in Lemmas 1 and 2, to compute results for the final entropy of certain of those automata listed in classes A and B of Table V, namely, for those not exclulded by Theorem 3; and for those in class C having the form $Q + \chi + \iota$ and $Q + \theta + \chi + \iota$. For $t \geqslant a(n)$, from (4.1)

$$S(t \geqslant a(n)) = -n^{-1} \log_2 N^{(c)} \tag{4.8}$$

where $N^{(c)}$ is the number of states which are fixed points or lie on shift cycles. This is determined from Lemmas 2 and 3, with the addition of 1 to include the fixed point $\mathbf{0}$, and another 1 if $\mathbf{1}$ is also a fixed point.

Two examples will illustrate this. From the Appendix, rule 184 has decompositions $\sigma^{-1}+\beta^{+}+\beta^{-}$ and $\sigma+\eta^{+}+\eta^{-}$. Thus, on states having only isolated 1's this rule acts as a right shift, while on states having only isolated 0's it acts as a left shift. From the corollary to Lemma 3 the number of such states in each case is L_n-1. We also see that both **0** and **1** are fixed points and these are the only fixed points. Thus, the total number of states which are fixed points or are on cycles for this automaton rule is given by $2(L_n-1)+2=2L_n$ and the final entropy is given by $n^{-1}(\log_2 L_n+1)$, where L_n is the nth Lucas number. Since the initial entropy has been normalized to 1, the maximum change in entropy in this automaton's evolution is $1-n^{-1}(\log_2 L_n+1)$.

As our second example, rule 48 has decomposition $\sigma^{-1}+\beta^{+}+\theta$. Again both **0** and **1** are fixed points. In addition, all states having only isolated 1's separated by two or more 0's are on shift cycles. No other states are fixed points, on are on cycles. Thus the final entropy is $n^{-1}\log_2(N_n(1,2)+2)$ and the maximum reduction in entropy is $1-n^{-1}\log_2(N_n(1,2)+2)$, where $N_n(1,2)$ is the nth term in the sequence defined by the recursion relation of Lemma 3.

5. DISCUSSION

The purpose of this paper has been to introduce a natural formalism for one-dimensional nearest neighbor cellular automata in terms of a basis set of eight nonlinear operators, and to use this formalism to determine significant properties such as fixed points, shift cycles, and entropy-reducing properties. This approach can be extended to more general cases as well. It is only necessary to write out the appropriate set of basis operators, and to express additive operators of interest (e.g., shift and the identity) in terms of these basis operators. The only difficulty which arises is that the number of basis operators grows exponentially. For example, for a one-dimensional neighborhood of radius 2, or for a two-dimensional von Neumann neighborhood, there are $2^5=32$ basis operators and a total of 2^{32} evolution rules.

A general problem with this approach is that it is difficult to compute cycle behavior other than shift cycles because of the nonlinear nature of the formalism. That is, $Q^*(\mu+\mu')\neq Q^*(\mu)+Q^*(\mu')$, hence iteration formulas cannot be easily computed.

Jen[2] has shown that all cycles will be multiple shift cycles, so one approach may be to study equations of the form $Q^{*k}(\mu)=\sigma^r(\mu)$ in order to determine, for any given Q^*, the values of k and r for which nontrivial solutions exist.

Another possible approach which shows some promise is to consider neighborhoods of size $2r+1$ ($r=1, 2, \ldots$), defining nonlinear operator formalisms for each, and constructing an embedding of $2r+1$ neighborhood rules into $2(r+1)+1$ rules. Whether the payoff from such a program is worth the computational effort involved remains to be seen.

APPENDIX. LEGAL REPRESENTATIONS OF NONGENERATIVE AUTOMATA

We list canonical representations and legal decompositions for all nongenerative nearest neighbor automata defined over $\mathbf{Z}_2$. In each case the canonical representation is listed first. An asterisk at the far left indicates an additive automaton. The decimal designation for each automaton is given at the fat left. If the automaton is not self-conjugate, the decimal designation for its conjugate automaton is listed in the next column.

The set of all nearest neighbor generative automata over $\mathbf{Z}_2$ is obtained by adding κ to each to the representations of this table, and adding 1 to the corresponding decimal designations.

Automaton and conjugate automaton			Canonical representation and legal decompositions
*	0		
	2	16	$\eta^+, \sigma+\beta^-+\Theta+\chi, \delta+\beta^++\beta^-+\eta^-, \Delta+\eta^-+\chi+\iota$
	4		$\iota, I+\beta^++\beta^-+\chi, \Delta+\eta^++\eta^-+\chi$
	6	20	$\eta^++\iota, I+\beta^++\beta^-+\eta^++\chi, D+\beta^++\Theta, \Delta+\eta^-+\chi$
	8	64	$\beta^-, \delta+\beta^++\eta^++\eta^-$
	10	80	$\beta^-+\eta^+, \sigma+\Theta+\chi, \delta+\beta^++\eta^-$
	12	68	$\beta^-+\iota, I+\beta^++\chi, \Delta+\beta^-+\eta^++\eta^-+\chi$
	14	84	$\beta^-+\eta^++\iota, \sigma+\Theta+\chi+\iota, I+\beta^++\eta^++\chi, D+\beta^++\beta^-+\Theta, \Delta+\beta^-+\eta^-+\chi$
	16	2	$\eta^-, \sigma^{-1}+\beta^++\Theta+\chi, \delta+\beta^++\beta^-+\eta^+, \Delta+\eta^++\chi+\iota$
	18		$\eta^++\eta^-, \delta+\beta^++\beta^-, \Delta+\chi+\iota$
	20	6	$\eta^-+\iota, I+\beta^++\beta^-+\eta^-+\chi, D^-+\beta^-+\Theta, \Delta+\eta^++\chi$
	22		$\eta^++\eta^-+\iota, I+\beta^++\beta^-+\eta^++\eta^++\chi, \Delta+\chi$
	24	66	$\beta^-+\eta^-, \sigma^{-1}+\beta^++\beta^-+\Theta+\chi, D^-+\Theta+\iota, \delta+\beta^+\eta^+$
	26	82	$\beta^-+\eta^++\eta^-, \delta+\beta^+$
	28	70	$\beta^-+\eta^-+\iota, I+\beta^++\eta^-+\chi, D^-+\Theta, \Delta+\beta^-+\eta^++\chi$
	30	86	$\beta^-+\eta^++\eta^-+\iota, I+\beta^++\eta^++\eta^-+\chi, \Delta+\beta^-+\chi$
	32		$\Theta, \sigma+\beta^-+\eta^++\chi, \sigma^{-1}+\beta^++\eta^-+\chi$
	34	48	$\eta^++\Theta, \sigma+\beta^-+\chi, \sigma^{-1}+\beta^++\eta^++\eta^-+\chi$
	36		$\Theta+\iota, I+\beta^++\beta^-+\Theta+\chi, D+\beta^++\eta^+, D^-+\beta^-+\eta^-$
	38	42	$\eta^++\Theta+\iota, D+\beta^+, D^-+\beta^-+\eta^++\eta^-$
	40	96	$\beta^-+\Theta, \sigma+\eta^++\chi, \sigma^{-1}+\beta^++\beta^-+\eta^-+\chi, D^-+\eta^-+\iota$

Table continued

822/66/5-6-15

Automaton and conjugate automaton		Canonical representation and legal decompositions
42	112	$\beta^- + \eta^+ + \Theta, \sigma + \chi, \sigma^{-1} + \beta^+ + \beta^- + \eta^+ + \eta^- + \chi, D^- + \eta^+ + \eta^- + \iota$
44	100	$\beta^- + \Theta + \iota, \sigma + \eta^+ + \chi + \iota, I + \beta^+ + \Theta + \chi, D + \beta^+ + \beta^- + \eta^+, D^- + \eta^-$
46	116	$\beta^- + \eta^+ + \Theta + \iota, \sigma + \chi + \iota, D + \beta^+ + \beta^-, D^- + \eta^+ + \eta^-$
48	34	$\eta^- + \Theta, \sigma + \beta^- + \eta^+ + \eta^- + \chi, \sigma^{-1} + \beta^+ + \chi$
50		$\eta^+ + \eta^- + \Theta, \sigma + \beta^- + \eta^- + \chi, \sigma^{-1} + \beta^+ + \eta^+ + \chi, \delta + \beta^+ + \beta^- + \Theta,$ $\Delta + \Theta + \chi + \iota$
52	38	$\eta^- + \Theta + \iota, D + \beta^+ + \eta^+ + \eta^-, D^- + \beta^-$
54		$\eta^+ + \eta^- + \Theta + \iota, D + \beta^+ + \eta^-, D^- + \beta^- + \eta^+, \Delta + \Theta + \chi$
56	98	$\beta^- + \eta^- + \Theta, \sigma + \eta^+ + \eta^- + \chi, \sigma^{-1} + \beta^+ + \beta^- + \chi, D^- + \iota$
58	114	$\beta^- + \eta^+ + \eta^- + \Theta, \sigma + \eta^- + \chi, \sigma^{-1} + \beta^+ + \beta^- + \eta^+ + \chi, D^- + \eta^+ + \iota,$ $\delta + \beta^+ + \Theta$
* 60	102	$\beta^- + \eta^- + \Theta + \iota, D^-$
62	118	$\beta^- + \eta^+ + \eta^- + \Theta + \iota, \sigma + \eta^- + \chi + \iota, D + \beta^+ + \beta^- + \eta^-, D^- + \eta^+,$ $\Delta + \beta^- + \Theta + \chi$
64	8	$\beta^+, \delta + \beta^- + \eta^+ + \eta^-$
66	24	$\beta^+ + \eta^+, \sigma + \beta^+ + \beta^- + \Theta + \chi, D + \Theta + \iota, \delta + \beta^- + \eta^-$
68	12	$\beta^+ + \iota, I + \beta^- + \chi, \Delta + \beta^+ + \eta^+ + \eta^- + \chi$
70	28	$\beta^+ + \eta^+ + \iota, I + \beta^- + \eta^+ + \chi, D + \Theta, \Delta + \beta^+ + \eta^- + \chi$
72		$\beta^+ + \beta^-, I + \chi + \iota, \delta + \eta^+ + \eta^-$
74	88	$\beta^+ + \beta^- + \eta^+, \sigma + \beta^+ + \Theta + \chi, I + \eta^+ + \chi + \iota, \delta + \eta^-$
76		$\beta^+ + \beta^- + \iota, I + \chi, \delta + \eta^+ + \eta^- + \iota, \Delta + \beta^+ + \beta^- + \eta^+ + \eta^- + \chi$
78	92	$\beta^+ + \beta^- + \eta^+ + \iota, I + \eta^+ + \chi, D + \beta^- + \Theta, \delta + \eta^- + \iota, \Delta + \beta^+ + \beta^- + \eta^- + \chi$
80	10	$\beta^+ + \eta^-, \sigma^{-1} + \Theta + \chi, \delta + \beta^- + \eta^+$
82	26	$\beta^+ + \eta^+ + \eta^-, \delta + \beta^-$
84	14	$\beta^+ + \eta^- + \iota, \sigma^{-1} + \Theta + \chi + \iota, I + \beta^- + \eta^- + \chi, D^- + \beta^+ + \beta^- + \Theta,$ $\Delta + \beta^+ + \eta^+ + \chi$
86	30	$\beta^+ + \eta^+ + \eta^- + \iota, I + \beta^- + \eta^+ + \eta^- + \chi, \Delta + \beta^+ + \chi$
88	74	$\beta^+ + \beta^- + \eta^-, \sigma^{-1} + \beta^- + \Theta + \chi, I + \eta^- + \chi + \iota, \delta + \eta^+$
* 90		$\beta^+ + \beta^- + \eta^+ + \eta^-, \delta$
92	78	$\beta^+ + \beta^- + \eta^- + \iota, I + \eta^- + \chi, D^- + \beta^+ + \Theta, \delta + \eta^+ + \iota, \Delta + \beta^+ + \beta^- + \eta^+ + \chi$
94		$\beta^+ + \beta^- + \eta^+ + \eta^- + \iota, I + \eta^+ + \eta^- + \chi, \delta + \iota, \Delta + \beta^+ + \beta^- + \chi$
96	40	$\beta^+ + o, \sigma + \beta^+ + \beta^- + \eta^+ + \chi, \sigma^{-1} + \eta^- + \chi, D + \eta^+ + \iota$
98	56	$\beta^+ + \eta^+ + o, \sigma + \beta^+ + \beta^- + \chi, \sigma^{-1} + \eta^+ + \eta^- + \chi, D + \iota$
100	44	$\beta^+ + o + \iota, \sigma^{-1} + \eta^- + \chi + \iota, I + \beta^- + \Theta + \chi, D + \eta^+, D^- + \beta^+ + \beta^- + \eta^-$
* 102	60	$\beta^+ + \eta^+ + \Theta + \iota, D$
104		$\beta^+ + \beta^- + \Theta, \sigma + \beta^+ + \eta^+ + \chi, \sigma^{-1} + \beta^- + \eta^- + \chi, I + \Theta + \chi + \iota$
106	120	$\beta^+ + \beta^- + \eta^+ + \Theta, \sigma + \beta^+ + \chi, \sigma^{-1} + \beta^- + \eta^+ + \eta^- + \chi$
108		$\beta^+ + \beta^- + \Theta + \iota, I + \Theta + \chi, D + \beta^- + \eta^+, D^- + \beta^+ + \eta^-$
110	124	$\beta^+ + \beta^- + \eta^+ + \Theta + \iota\kappa, D + \beta^-, D^- + \beta^+ + \eta^+ + \eta^-$
112	42	$\beta^+ + \eta^- + \Theta, \sigma + \beta^+ + \beta^- + \eta^+ + \eta^- + \chi, \sigma^{-1} + \chi, D + \eta^+ + \eta^- + \iota$
114	58	$\beta^+ + \eta^+ + \eta^- + \Theta, \sigma + \beta^+ + \beta^- + \eta^- + \chi, \sigma^{-1} + \eta^+ + \chi, D + \eta^- + \iota, \delta + \beta^- + \Theta$
116	46	$\beta^+ + \eta^- + \Theta + \iota, \sigma^{-1} + \chi + \iota, D + \eta^+ + \eta^-, D^- + \beta^+ + \beta^-$
118	62	$\beta^+ + \eta^+ + \eta^- + \Theta + \iota, \sigma^{-1} + \eta^+ + \chi + \iota, D + \eta^-, D^- + \beta^+ + \beta^- + \eta^+,$ $\Delta + \beta^+ + \Theta + \chi$

Table continued

Automaton and conjugate automaton	Canonical representation and legal decompositions
120 106	$\beta^{+}+\beta^{-}+\eta^{-}+\Theta, \sigma+\beta^{+}+\eta^{+}+\eta^{-}+\chi, \sigma^{-1}+\beta^{-}+\chi$
122	$\beta^{+}+\beta^{-}+\eta^{+}+\eta^{-}+\Theta, \sigma+\beta^{+}+\eta^{-}+\chi, \sigma^{-1}+\beta^{-}+\eta^{+}+\chi, \delta+\Theta$
124 110	$\beta^{+}+\beta^{-}+\eta^{-}+\Theta+\iota, D+\beta^{-}+\eta^{+}+\eta^{-}, D^{-}+\beta^{+}$
126	$\beta^{+}+\beta^{-}+\eta^{+}+\eta^{-}+\Theta+\iota, D+\beta^{-}+\eta^{-}, D^{-}+\beta^{+}+\eta^{+}, \delta+\Theta+\iota,$ $\Delta+\beta^{+}+\beta^{-}+\chi+\Theta$
128	$\chi, \delta+\beta^{+}+\beta^{-}+\eta^{+}+\eta^{-}+\chi, \Delta+\eta^{+}+\eta^{-}+\iota$
130 144	$\eta^{+}+\chi, \sigma+\beta^{-}+\Theta, \delta+\beta^{+}+\beta^{-}+\eta^{-}+\chi, \Delta+\eta^{-}+\iota$
132	$\chi+\iota, I+\beta^{+}+\beta^{-}, \Delta+\eta^{+}+\eta^{-}$
134 148	$\eta^{+}+\chi+\iota, I+\beta^{+}+\beta^{-}+\eta^{+}, D+\beta^{+}+\Theta+\chi, \Delta+\eta^{-}$
136 192	$\beta^{-}+\chi, \delta+\beta^{+}+\eta^{+}+\eta^{-}+\chi$
138 208	$\beta^{-}+\eta^{+}+\chi, \sigma+\Theta, \delta+\beta^{+}+\eta^{-}+\chi$
140 196	$\beta^{-}+\chi+\iota, I+\beta^{+}, \Delta+\beta^{-}+\eta^{+}+\eta^{-}$
142 212	$\beta^{-}+\eta^{+}+\chi+\iota, \sigma+\Theta+\iota, I+\beta^{+}+\eta^{+}, D+\beta^{+}+\beta^{-}+\Theta+\chi, \Delta+\beta^{-}+\eta^{-}$
144 130	$\eta^{-}+\chi, \sigma^{-1}+\beta^{+}+\Theta, \delta+\beta^{+}+\beta^{-}+\eta^{+}+\chi, \Delta+\eta^{+}+\iota$
146	$\eta^{+}+\eta^{-}+\chi, \delta+\beta^{+}+\beta^{-}+\chi, \Delta+\iota$
148 134	$\eta^{-}+\chi+\iota, I+\beta^{+}+\beta^{-}+\eta^{-}, D^{-}+\beta^{-}+\Theta+\chi, \Delta+\eta^{+}$
* 150	$\eta^{+}+\eta^{-}+\chi+\iota, \Delta$
152 194	$\beta^{-}+\eta^{-}+\chi, \sigma^{-1}+\beta^{+}+\beta^{-}+\Theta, D^{-}+\Theta+\chi+\iota, \delta+\beta^{+}+\eta^{+}+\chi$
154 210	$\beta^{-}+\eta^{+}+\eta^{-}+\chi, \delta+\beta^{+}+\chi$
156 198	$\beta^{-}+\eta^{-}+\chi+\iota, I+\beta^{-}+\eta^{+}, D^{-}+\Theta+\chi, \Delta+\beta^{-}+\eta^{+}$
158 214	$\beta^{-}+\eta^{+}+\eta^{-}+\chi+\iota, I+\beta^{+}+\eta^{+}+\eta^{-}, \Delta+\beta^{-}$
160	$\Theta+\chi, \sigma+\beta^{-}+\eta^{+}, \sigma^{-1}+\beta^{+}+\eta^{-}$
162 176	$\eta^{+}+\Theta+\chi, \sigma+\beta^{-}, \sigma^{-1}+\beta^{+}+\eta^{+}+\eta^{-}$
164	$\Theta+\chi+\iota, I+\beta^{+}+\beta^{-}+\Theta, D+\beta^{+}+\eta^{+}+\chi, D^{-}+\beta^{-}+\eta^{-}+\chi$
166 180	$\eta^{+}+\Theta+\chi+\iota, D+\beta^{+}+\chi, D^{-}+\beta^{-}+\eta^{+}+\eta^{-}+\chi$
168 224	$\beta^{-}+\Theta+\chi, \sigma+\eta^{+}, \sigma^{-1}+\beta^{+}+\beta^{-}+\eta^{-}, D^{-}+\eta^{-}+\chi+\iota$
* 170 240	$\beta^{-}+\eta^{+}+\Theta+\chi, \sigma$
172 228	$\beta^{-}+\Theta+\chi+\iota, \sigma+\eta^{+}+\iota, I+\beta^{+}+\Theta, D+\beta^{+}+\beta^{-}+\eta^{+}+\chi, D^{-}+\eta^{-}+\chi$
174 244	$\beta^{-}+\eta^{+}+\Theta+\chi+\iota, \sigma+\iota, D+\beta^{+}+\beta^{-}+\chi, D^{-}+\eta^{+}+\eta^{-}+\chi$
176 162	$\eta^{-}+\Theta+\chi, \sigma+\beta^{-}+\eta^{+}+\eta^{-}, \sigma^{-1}+\beta^{+}$
178	$\eta^{+}+\eta^{-}+\Theta+\chi, \sigma+\beta^{-}+\eta^{-}, \sigma^{-1}+\beta^{+}+\eta^{+}, \delta+\beta^{+}+\beta^{-}+\Theta+\chi, \Delta+\Theta+\iota$
180 166	$\eta^{-}+\Theta+\chi+\iota, D+\beta^{+}+\eta^{+}+\eta^{-}+\chi, D^{-}+\beta^{-}+\chi$
182	$\eta^{+}+\eta^{-}+\Theta+\chi+\iota, D+\beta^{+}+\eta^{-}+\chi, D^{-}+\beta^{-}+\eta^{+}+\chi, \Delta+\Theta$
184 226	$\beta^{-}+\eta^{-}+\Theta+\chi, \sigma+\eta^{+}+\eta^{-}, \sigma^{-1}+\beta^{+}+\beta^{-}, D^{-}+\chi+\iota$
186 242	$\beta^{-}+\eta^{+}+\eta^{-}\mathrm{w}+\Theta+\chi, \sigma+\eta^{-}, \sigma^{-1}+\beta^{+}+\beta^{-}+\eta^{+}, D^{-}+\eta^{+}+\chi+\iota$
188 230	$\beta^{-}+\eta^{-}+\Theta+\chi+\iota, \sigma+\eta^{+}+\eta^{-}+\iota, D+\beta^{+}+\beta^{-}+\eta^{+}+\eta^{-}+\chi, D^{-}+\chi$
190 246	$\beta^{-}+\eta^{+}+\eta^{-}+\Theta+\chi+\iota, \sigma+\eta^{-}+\iota, D+\beta^{+}+\beta^{-}+\eta^{-}+\chi, D^{-}+\eta^{+}+\chi,$ $\Delta+\beta^{-}+\Theta$
192 136	$\beta^{+}+\chi, \delta+\beta^{-}+\eta^{+}+\eta^{-}+\chi$
194 152	$\beta^{+}+\eta^{+}+\chi, \sigma+\beta^{+}+\beta^{-}+\Theta, D+\Theta+\chi+\iota, \delta+\beta^{-}+\eta^{-}+\chi$
196 140	$\beta^{+}+\chi+\iota, I+\beta^{-}, \Delta+\beta^{+}+\eta^{+}+\eta^{-}$
198 156	$\beta^{+}+\eta^{-}+\chi+\iota, I+\beta^{+}+\eta^{-}, D+\Theta+\chi, \Delta+\beta^{+}+\eta^{-}$
200	$\beta^{+}+\beta^{-}+\chi, I+\iota, \delta+\eta^{+}+\eta^{-}+\chi$
202 216	$\beta^{+}+\beta^{-}+\eta^{+}+\chi, \sigma+\beta^{+}+\Theta, I+\eta^{+}+\iota, \delta+\eta^{-}+\chi$

Table continued

Automaton and conjugate automaton	Canonical representation and legal decompositions
* 204	$\beta^+ + \beta^- + \chi + \iota, I$
206 220	$\beta^+ + \beta^- + \eta^+ + \chi + \iota, I + \eta^+, D + \beta^- + \Theta + \chi, \delta + \eta^- + \chi + \iota, \Delta + \beta^+ + \beta^- + \eta^-$
208 138	$\beta^+ + \eta^- + \chi, \sigma^{-1} + \Theta, \delta + \beta^- + \eta^+ + \chi$
210 154	$\beta^+ + \eta^+ + \eta^- + \chi, \delta + \beta^- + \chi$
212 142	$\beta^+ + \eta^- + \chi + \iota, \sigma^{-1} + \Theta + \iota, I + \beta^- + \eta^-, D^- + \beta^+ + \beta^- + \Theta + \chi, \Delta + \beta^+ + \eta^+$
214 158	$\beta^+ + \eta^+ + \eta^- + \chi + \iota, I + \beta^- + \eta^+ + \eta^-, \Delta + \beta^+$
216 202	$\beta^+ + \beta^- + \eta^- + \chi, \sigma^{-1} + \beta^- + \Theta, I + \eta^- + \iota, \delta + \eta^+ + \chi$
218	$\beta^+ + \beta^- + \eta^+ + \eta^- + \chi, I + \eta^+ + \eta^- + \iota, \delta + \chi$
220 206	$\beta^+ + \beta^- + \eta^- + \chi + \iota, I + \eta^-, D^- + \beta^+ + \Theta + \chi, \delta + \eta^+ + \chi + \iota,$ $\Delta + \beta^+ + \beta^- + \eta^+$
222	$\beta^+ + \beta^- + \eta^+ + \eta^- + \chi + \iota, I + \eta^+ + \eta^-, \delta + \chi + \iota, \Delta + \beta^+ + \beta^-$
224 168	$\beta^+ + \Theta + \chi, \sigma + \beta^+ + \beta^- + \eta^+, \sigma^{-1} + \eta^-, D + \eta^+ + \chi + \iota$
226 184	$\beta^+ + \eta^+ + \Theta + \chi, \sigma + \beta^+ + \beta^-, \sigma^{-1} + \eta^+ + \eta^-, D + \chi + \iota$
228 172	$\beta^+ + \Theta + \chi + \iota, \sigma^{-1} + \eta^- + \iota, I + \beta^- + \Theta, D + \eta^+ + \chi, D^- + \beta^+ + \beta^- + \eta^- + \chi$
230 188	$\beta^+ + \eta^+ + \Theta + \chi + \iota, \sigma^{-1} + \eta^+ + \eta^- + \iota, D + \chi, D^- + \beta^+ + \beta^- + \eta^+ + \eta^- + \chi$
232	$\beta^+ + \beta^- + \Theta + \chi, \sigma + \beta^+ + \eta^+, \sigma^{-1} + \beta^- + \eta^-, I + \Theta + \iota$
234 248	$\beta^+ + \beta^- + \eta^+ + \Theta + \chi, \sigma + \beta^+, \sigma^{-1} + \beta^- + \eta^+ + \eta^-$
236	$\beta^+ + \beta^- + \Theta + \chi + \iota, I + \Theta, D + \beta^- + \eta^+ + \chi, D^- + \beta^+ + \eta^- + \chi$
238 252	$\beta^+ + \beta^- + \eta^+ + \Theta + \chi + \iota, D + \beta^- + \chi, D^- + \beta^+ + \eta^+ + \eta^- + \chi$
* 240 170	$\beta^+ + \eta^- + \Theta + \chi, \sigma^{-1}$
242 186	$\beta^+ + \eta^+ + \eta^- + \Theta + \chi, \sigma + \beta^+ + \beta^- + \eta^-, \sigma^{-1} + \eta^+, D + \eta^- + \chi + \iota$
244 174	$\beta^+ + \eta^- + \Theta + \chi + \iota, \sigma^{-1} + \iota, D + \eta^+ + \eta^- + \chi, D^- + \beta^+ + \beta^- + \chi$
246 190	$\beta^+ + \eta^+ + \eta^- + \Theta + \chi + \iota, \sigma^{-1} + \eta^+ + \iota, D^- + \beta^+ + \eta^+ + \chi, D + \eta^- + \chi,$ $\Delta + \beta^+ + \Theta$
248 234	$\beta^+ + \beta^- + \eta^- + \Theta + \chi, \sigma + \beta^+ + \eta^+ + \eta^-, \sigma^{-1} + \beta^-$
250	$\beta^+ + \beta^- + \eta^+ + \eta^- + \Theta + \chi, \sigma + \beta^+ + \eta^-, \sigma^{-1} + \beta^- + \eta^+, \delta + \Theta + \chi$
252 238	$\beta^+ + \beta^- + \eta^- + \Theta + \chi + \iota, D + \beta^- + \eta^+ + \eta^+ + \chi, D^- + \beta^+ + \chi$
254	$\beta^+ + \beta^- + \eta^+ + \eta^- + \Theta + \chi + \iota, D + \beta^- + \eta^- + \chi, D^- + \beta^+ + \eta^+ + \chi,$ $\delta + \Theta + \chi + \iota, \Delta + \beta^+ + \beta^- + \Theta$

ACKNOWLEDGMENTS

Proof of Lemmas 1 and 2 arose in a conversation with Al Weiss. I also thank a referee who provided a detailed and very useful critique of an earlier manuscript. This work was supported by NSERC Operating Grant OGP 0024817.

REFERENCES

1. O. Martin, A. M. Odlyzko, and S. Wolfram, *Commun. Math. Phys.* **93**:219–258 (1984).
2. E. Jen, *Commun. Math. Phys.* **118**:569–590 (1988).
3. B. Voorhees, *Physica D* **45**:26–35 (1990).
4. S. Wolfram, *Rev. Mod. Phys.* **55**:601–644 (1983).
5. D. A. Lind, *Physica D* **10**:36–44 (1984).
6. B. Voorhees, *Int. J. Theor. Phys.* **28**:1387–1396 (1989).